Uniqueness of free actions of finite abelian groups on surfaces

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Outline

Abstract

Let G be a finite abelian group. We list all the cases where the conjugacy class of orientation-preserving free G-actions on a closed surface is unique. A joint work with Y. She.

Double action v.s. Topological equivalence

Two homomorphisms η, η' are said to be equivalent under the double action if

$$\begin{array}{ccc} \Lambda & \xrightarrow{\eta} & G \\ \downarrow \xi & & \downarrow \zeta \\ \Lambda & \xrightarrow{\eta'} & G \end{array}$$

Here, $(\zeta, \xi) \in \mathrm{Aut}(G) \times \mathrm{Aut}(\Lambda)$.

Topological equivalence of G-actions

$$\begin{array}{ccc} G \times X \ni (g, x) & \longrightarrow & g(x) \in X \\ \downarrow & & \downarrow \\ G \times X \ni (\zeta(g), f(x)) & \longrightarrow & \zeta(g)(f(x)) = f(g(x)) \end{array}$$

i.e. $f^{-1}\zeta(g)f = g \in Home(X)$.

Basic relations

Given a compact surface S and a finite group G, there exist one-to-one correspondences among following three sets:

(1) topological equivalence classes of G-actions on S,

(2) equivalence classes of $\mathrm{Aut}(G) \times \mathrm{Aut}(\Lambda)$-actions of the set of epimorphisms from some Non-euclidean crystallographic group (NEC group) Λ to the group G, having fundamental group $\pi_1(S)$ as kernels, i.e. $1 \to \pi_1(S) \to \Lambda \to G \to 1$.

(3) conjugacy classes $[G]$ of subgroups of the mapping class group of S.

Very non-trivial relations: (2) to (1): Macbeath (1967). (3) to (1): Kerckhoff (1983).

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non-Euclidean crystallographic group (NEC group) Λ

(♠) Definition: discrete, cocompact subgroup of $Iso(\mathbb{H}^2)$.

(♡) Signature (notation):

$$\Lambda = (g; +/-; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

(♢) generators:

$$\begin{array}{ll} x_i \text{ (rotation)}, & 1 \leq i \leq r, \\ c_{ij}\text{(reflection)}, \; e_i, & 1 \leq i \leq k, \; 0 \leq j \leq s_i, \\ a_i, b_i\text{(translations)} & 1 \leq i \leq g. \\ d_i \text{ (glide reflection)}, & \end{array}$$

(♣) relations:

$$\begin{array}{ll} x_i^{m_i} = 1, & 1 \leq i \leq r, \\ c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1, & 1 \leq i \leq k, \; 0 \leq j \leq s_i, \\ c_{is_i} = e_i c_{i0} e_i^{-1}, & 1 \leq i \leq k, \\ x_1 \cdots x_r e_1 \cdots e_k [a_1, b_1] \cdots [a_g, b_g] = 1, & \end{array}$$

Automorphism groups of NEC's and braid groups

Let $\Lambda = (g; +; [m_1, \ldots, m_r])$. Then $\mathrm{Aut}(\Lambda)$ is generated by

- Dehn twist,
- $\sigma_{ij}(x_i \mapsto x_i x_j x_i^{-1}, x_j \mapsto x_i)$ for $r_i = r_j$, i.e. x_i and x_j have the same order,
- $\mu_{ij}(a_i \mapsto a_i u x_j u)$,
- $\nu_{ij}(b_i \mapsto b_i v x_j v)$.

i.e. restricted braid groups on surfaces.

Question: How many equivalence classes of $\mathrm{Aut}(G) \times \mathrm{Aut}(\Lambda)$-actions for epimorphisms from Λ to G (having surface group kernels iff x_i and its image has the same order)?

Our case: free action

(1) topological equivalence classes of free G-actions on S,

(2) equivalence classes of double-actions of the set of epimorphisms from $\Lambda = \pi_1(S')$ to the group G, having the fundamental group $\pi_1(S)$ as kernels.

Note:

(1) The kernel of such an epimorphism must be the fundamental group of a surface,

(2) $\chi(S) = |G| \cdot \chi(S')$

Homomorphisms and matrices

Since any homomorphism from $\pi_1(S_h)$ to a finite abelian group G factors through $H_1(S_h) = \mathbb{Z}^{2h}$, any homomorphism $\eta : \pi_1(S_h) \to G$ is determined by a matrix $M_{s\times(2h)}$, meaning

$$(\eta(\bar{a}_1), \eta(\bar{b}_1), \ldots, \eta(\bar{a}_h), \eta(\bar{b}_h)) = (\omega_1, \ldots, \omega_s)M,$$

In this sense,
an $\mathrm{Aut}(\pi_1(S_h))$-action is said to be a right action,
while an $\mathrm{Aut}(G)$-action is said to be a left action.

$$M \sim M' \ \Leftrightarrow M' = A\,M\,B$$

where $A \in \mathrm{Aut}(G)$, $B \in \mathrm{Sp}(2h, \mathbb{Z})$.

Right action (generators of symplectic matrix group)

(1) $$L_{2i-1,2i} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 1 & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$, indicating $\mathfrak{A}_i : \bar{b}_i \mapsto \bar{b}_i + \bar{a}_i$;

(2) $$L_{2i,2i-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & 1 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$, indicating $\mathfrak{B}_i : \bar{a}_i \mapsto \bar{a}_i + \bar{b}_i$;

(3) $$L^{-1}_{2i,2i+2} L_{2i+1,2i-1} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & 0 & 0 & 0 & & \\ & & 0 & 1 & 0 & -1 & & \\ & & 1 & 0 & 1 & 0 & & \\ & & 0 & 0 & 0 & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$, indicating

$\mathfrak{Z}_i : \bar{a}_i \mapsto \bar{a}_i + \bar{a}_{i+1}, \bar{b}_{i+1} \mapsto \bar{b}_{i+1} - \bar{b}_i$;

Right action (generators of symplectic matrix group)

(4) $T_{2i-1,2i}D_{2i}(-1) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$, indicating

$\mathfrak{R}_i : \bar{a}_i \mapsto \bar{b}_i, \bar{b}_i \mapsto -\bar{a}_i;$

(5) $T_{2i-1,2i+1}T_{2i,2i+2} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & 0 & 1 & 0 & & \\ & & 0 & 0 & 0 & 1 & & \\ & & 1 & 0 & 0 & 0 & & \\ & & 0 & 1 & 0 & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$, indicating

$\mathfrak{S}_i : \bar{a}_i \mapsto \bar{a}_{i+1}, \bar{b}_i \mapsto \bar{b}_{i+1}, \bar{a}_{i+1} \mapsto \bar{a}_i, \bar{b}_{i+1} \mapsto \bar{b}_i.$

What is the left action

Any matrix A from left is determined by an element in $\mathrm{Aut}(G)$. (Assumption: $G = Z_{n_1} \oplus \cdots \oplus Z_{n_s}$ with $n_s | \cdots | n_1$).

The generators of $\mathrm{Aut}(G)$ are: $L_{i,j}$ for $i > j$, $L_{i,j}^{n_i/n_j}$ for $i < j$, and $D_t(k)$ with $gcd(k, n_t) = 1$, ...

The structure of $\mathrm{Aut}(G)$ is not simple.

An example

When G is a cyclic group.

$$(*,*,*,*) \overset{\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{B}_1,\mathfrak{B}_2}{\mapsto} (d_1, d_1, d_2, 0) \overset{\mathfrak{Z}_1}{\mapsto} (d_1 + d_2, d_1, d_2, -d_1 + d_2) \overset{\mathfrak{Z}_1,\cdots}{\mapsto} (0, d, 0, d) \overset{\mathfrak{Z}_1}{\mapsto} (0, d, 0, 0).$$

d_1 is the GCD of first two entries, d_2 is the GCD of last two entries, and d is the GCD of all entries.

A folk Theorem: the free action of cyclic group on a close surface is unique up to conjugation.

Case $G = Z_{n_1} \oplus Z_{n_2}$

Any epimorphism from $\pi_1(S_h)$ to G is double-equivalent to one determined by $\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & s & t & 0 & \cdots \end{pmatrix} \overset{\mathfrak{Z}_1}{\mapsto} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ t & s & t & -s & \cdots \end{pmatrix}$, to $\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & q & 1 & 0 & \cdots \end{pmatrix}$, with $q|n_2$.

Note that $\mathfrak{W}^1(\eta_q)$ has order n_2/q. We obtain that

The number of conjugacy classes of free action of $G = Z_{n_1} \oplus Z_{n_2}$ on closed surface is the same as the number of factor of n_2.

Some invariants from cohomology

Lemma

Let $\eta : \pi_1(S_h) \to G$ be a homomorphism. Then for each positive integer k,

$$\mathfrak{W}^k(\eta) =: \sum_{1 \le i_1 < \cdots i_k \le s} \eta(\bar{a}_{i_1}) \wedge \eta(\bar{b}_{i_1}) \wedge \cdots \wedge \eta(\bar{a}_{i_k}) \wedge \eta(\bar{b}_{i_k}) \in \wedge^{2k} G$$

is an $\mathrm{Aut}(\pi_1(S_h))$-action invariant.

Corollary: the order $|\mathfrak{W}^k(\eta)|$ is a double-action invariant.

Any homomorphism η from $\pi_1(S_h)$ to G can be considered as a element in $H^1(S_h, G)$. For $k = 1$, $\mathfrak{W}^1(\eta) = (\eta \cup \eta)[S_h] \in G \otimes G$, an invariant given by S. A. Broughton and A. Wootton (2007).

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One more example

Proposition

If $n_1 \geq n_2 = n_3 \geq n_4$ and $\gcd(\frac{n_1}{n_2}, \frac{n_2}{n_4}) = 1$, then the double-equivalence class of epimorphisms from $\pi_1(S_2)$ to $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3} \oplus \mathbb{Z}_{n_4}$ is unique.

Begin with $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{1,2} & 1 & 0 \\ 0 & m_{1,3} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Since $\gcd(\frac{n_1}{n_2}, \frac{n_2}{n_4}) = 1$, there are two integers v and w such that $v\frac{n_1}{n_2} + w\frac{n_2}{n_4} = 1$. Thus, $m_{1,j} = m_{1,j}(v\frac{n_1}{n_2} + w\frac{n_2}{n_4})$ for $j = 2, 3$. We have that

$$M \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{1,2}(1 - w\frac{n_2}{n_4}) & 1 & 0 \\ 0 & m_{1,3}(1 - w\frac{n_2}{n_4}) & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{1,2}v\frac{n_1}{n_2} & 1 & 0 \\ 0 & m_{1,3}v\frac{n_1}{n_2} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We write $u' = \frac{un_1}{n_2}$ and $t' = \frac{tn_1}{n_2}$. Then

$$\xrightarrow{T_{3,4}D_4(-1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u' & 0 & -1 \\ 0 & t' & 1 & 0 \end{pmatrix} \xrightarrow{T_{1,3}T_{2,4}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & u' \\ 1 & 0 & 0 & t' \end{pmatrix}$$

$$\xrightarrow{(L_{2,4}^{-1}L_{3,1})^{-u'}} \begin{pmatrix} -u' & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & t' \end{pmatrix} \xrightarrow[(L_{1,3})^{u'}]{} \begin{pmatrix} 0 & 0 & 1 & u't' \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & t' \end{pmatrix}$$

$$\xrightarrow{(L_{3,4})^{-u't'}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & t' \end{pmatrix} \xrightarrow{(T_{1,2}D_2(-1))^{-1}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t' \end{pmatrix}$$

$$\xrightarrow{(L_{2,4}^{-1}L_{3,1})^{t'}} \begin{pmatrix} t' & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[(L_{1,2})^{-t'}]{} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{T_{1,3}T_{2,4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The main result

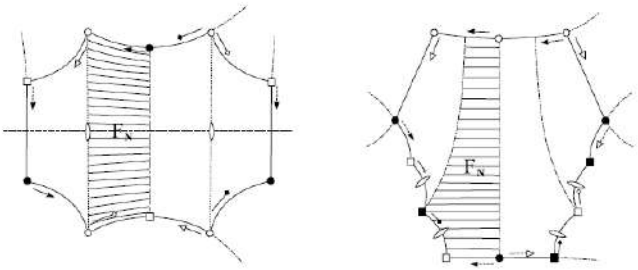
Theorem

Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ *be a finite abelian group. The the* $\mathrm{Aut}(G) \times \mathrm{Aut}(\pi_1(S_h))$*-equivalence class is unique if and only if either*
(1) $s = 1$, *or*
(2) $s = 2h - 1$, $n_2 = \cdots = n_{2h-1}$ *and* $\gcd(\frac{n_1}{n_2}, n_2) = 1$, *or*
(3) $s = 2h$, $n_2 = \cdots = n_{2h-1}$ *and* $\gcd(\frac{n_1}{n_2}, \frac{n_2}{n_{2h}}) = 1$.

Theorem

Assume that G is not a cyclic group, i.e. $s > 1$. *The closed surface* S_g *admits a unique free G-action if and only if* $g = n_1(h-1)n^{2h-2}t + 1$ *for some integers* h, n *and* $n = n_2 = \cdots = n_{2h-1}$, *where integers* n, t *satisfy one of the following conditions:*

Tilling: Geometric interpretation



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Thank you for your paying attention.